To solve equation (7), we compute $\frac{3b}{2a}\sqrt{\frac{3}{a}}$, treat it as a sine, and compute the corresponding angle—which we call $3\phi_1$—by (5) and (6), taking the real part of 3ϕ between $-90°$ and $+90°$. Other possible values of 3ϕ are $3\phi_2=3\phi_1+360°$, $3\phi_3=3\phi_1+720°$. The roots are therefore, by (10),

$$\left.\begin{aligned} x_1&=2\sqrt{\frac{a}{3}}\sin\phi_1 \\ x_2&=2\sqrt{\frac{a}{3}}\sin\phi_2=2\sqrt{\frac{a}{3}}\sin(\phi_1+120°) \\ x_3&=2\sqrt{\frac{a}{3}}\sin\phi_3=2\sqrt{\frac{a}{3}}\sin(\phi_1+240°) \end{aligned}\right\} \qquad (12)$$

When a and b are restricted to real values, we readily find the well-known conditions for the reality of all roots:

$$a>0 \text{ and } 27b^2<4a^3.$$

The condition for equal roots holds whether the coefficients are real or complex. By introducing the Gudermannian angle θ, defined by $\phi=\log_e\tan(45°+\frac{1}{2}\theta)$, from which follows that

$$\text{Cosh}\phi=\sec\theta, \quad \text{Sinh}\phi=\tan\theta, \quad \text{etc.},$$

we can reduce the general formulas given above to the special ones involving tangents of auxiliary angles sometimes used in the case of a single real root.

It is worth noting that a table of Mercator's parts—calculated for a spherical earth—is a table of inverse Gudermannians, and may be used to give the hyperbolic functions if a regular table of them is not at hand. A table of Mercator's parts carried to $\frac{1}{1000}$ minute is given in Callet's "Tables Nautiques." The second alternative form under (5) and (6) is generally the more convenient for numerical work with complex coefficients. The inner radical represents the side of a triangle whose other sides are r^2 and 1, and their included angle a. There are small tables for the solution of this case, but in view of its frequent occurence either directly, or in an equivalent form, it seems rather remarkable that no extensive tables for it are in general use.

Example: Solve $x^3+ix+1+i=0$.

Here $a=-i$, $b=1+i$, $\sqrt{\frac{a}{3}}=\frac{1-i}{\sqrt{6}}$, $\sqrt{\frac{3}{a}}=(1+i)\sqrt{\frac{3}{2}}$, $\sin3\phi=-\frac{3}{2}\sqrt{6}$.

From (5) and (6), since $p=-\frac{3}{2}\sqrt{6}$ and $q=0$, $u=\sin^{-1}(-1)=-90°$, $v=\text{Sinh}^{-1}(\frac{5}{2}\sqrt{2})=1.97544$, $\text{Sinh}\frac{v}{3}=0.70709$, $\text{Cosh}\frac{v}{3}=1.22473$;

$$x_1=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times 1.22473+i\frac{\sqrt{3}}{2}\times 0.70709)=-0.00001+i\times 0.99997,$$

$$x_2=\frac{2(1-i)}{\sqrt{6}}(1.22473)=0.99998(1-i),$$

$$x_3=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times 1.22473-i\frac{\sqrt{3}}{2}\times 0.70709)=-0.99997+i\times 0.00001.$$

These are from five-figure tables; the exact roots are i, $1-i$, and -1.

ON THE EXPANSION OF DEVERTEBRATED THREE DIMENSIONAL DETERMINANTS AND THE EXTENSION OF CAYLEY'S EXPANSION THEOREM.

By ORLANDO S. STETSON, Syracuse University.

The primary object of this paper is to extend to three dimensional or cubical determinants* the general law for the expansion of a two dimensional devertebrated determinant† in terms of the elements of the principal diagonal of the given determinant and their co-axial minors or, in other words, to develop a general law for the expansion of a devertebrated cubical determinant in terms of the elements of its principal diagonal plane and their co-planar minors.

The secondary object is to show how Cayley's Expansion Theorem may be extended to cubical determinants with but slight modifications. A formula for the number of terms of a cubical determinant which are independent of the elements of the principal diagonal plane is also given.

Let $\triangle$ denote a cubical determinant of order n and let $\triangle'_a$ denote a new cubical determinant formed by adding a different variable to each of the n^2 elements of the principal diagonal plane.

Expanding $\triangle'_a$ as a cubical determinent with binomial elements, the term independent of the variable will be the given cubical determinant $\triangle$. The next n^2 terms are the products of the respective variables and their corresponding cubical minors of order $n-1$; similarly, the next $\frac{n^2(n-1)^2}{2!}$ terms are products

*An exposition and bibliography of cubical determinants is given by E. R. Hedrick, *Annals of Mathematics*, Ser. 2, Vol. 1 (1900), pp. 49–67.

†Stetson, MONTHLY, Vol. XI (1904), pp. 166–168.

of pairs of variables of the principal diagonal plane and their corresponding cubical minors of order $n-2$, no two of the variables being common to the same plane, and so on.

Now, let us assume any α elements of the principal diagonal plane of the given cubical determinant to be equal to zero and let us replace each of the corresponding variables by the negative of the element, placing the remaining $n^2-\alpha$ variables equal to zero; the general law may then be stated as follows:

A devertebrated cubical determinant of order n containing α zero elements in its principal diagonal plane may be developed into a series of terms of which the first is the given cubical determinant, $\triangle$; the next α being the products of each of the α elements into its co-planar cubical minor of order $n-1$; the next set consisting of the products of every possible pair of the α elements of the principal diagonal plane and its coplanar cubical minor of order $n-2$, no two of the elements being common to a same plane, and so on; the signs of the respective terms are positive or negative according as the combinations of the α elements are even or odd.

By a proper selection of the α elements this general law may be seen to contain as a particular case the law for the expansion of a devertebrated cubical determinant in which the α zero elements lie along the principal diagonal of the ordinary determinant of the principal diagonal plane.

For $\alpha=n^2$, the expansion will be seen to contain, also, the law for the expression of an invertebrate coplanar minor in terms of the vertebrate coplanar minors and the elements of the principal diagonal plane.

As illustrations, we may assume a cubical determinant, $\triangle'_\alpha$ of order 3 in which the elements in parentheses designate the variables which are added to the elements of the principal diagonal plane. We will write the sheets under one another, thus—

$$\triangle'_\alpha = \left\{ \begin{array}{lll} \left\{ \begin{array}{lll} 111+(111) & 112 & 113 \\ 121 & 122+(122) & 123 \\ 131 & 132 & 133+(133) \end{array} \right. & & \\ \left\{ \begin{array}{lll} 211+(211) & 212 & 213 \\ 221 & 222+(222) & 223 \\ 231 & 232 & 233+(233) \end{array} \right. & & \\ \left\{ \begin{array}{lll} 311+(311) & 312 & 313 \\ 321 & 322+(322) & 323 \\ 331 & 332 & 333+(333) \end{array} \right. & & \end{array} \right.$$

Expanding $\triangle'_\alpha$ as a cubical determinant with binomial elements, the expansion may be written in the following form—

$$\begin{aligned} \triangle'_\alpha = \triangle\ &+(111)[111]+(122)[122]+(133)[133]+(211)[211]+(222)[222] \\ &+(233)[233]+(311)[311]+(322)[322]+(333)[333] \\ &+(111)(222)[111,\ 222]+(111)(322)[111,\ 322]+(211)(122) \end{aligned}$$

From (5) and (6), since $p=-\frac{3}{2}\sqrt{6}$ and $q=0$, $u=\sin^{-1}(-1)=-90°$, $v=\text{Sinh}^{-1}(\frac{5}{2}\sqrt{2})=1.97544$, $\text{Sinh}\frac{v}{3}=0.70709$, $\text{Cosh}\frac{v}{3}=1.22473$;

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Let $\triangle$ denote a cubical determinant of order n and let $\triangle'_a$ denote a new cubical determinant formed by adding a different variable to each of the n^2 elements of the principal diagonal plane.

Expanding $\triangle'_a$ as a cubical determinent with binomial elements, the term independent of the variable will be the given cubical determinant $\triangle$. The next n^2 terms are the products of the respective variables and their corresponding cubical minors of order $n-1$; similarly, the next $\frac{n^2(n-1)^2}{2!}$ terms are products

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of pairs of variables of the principal diagonal plane and their corresponding cubical minors of order $n-2$, no two of the variables being common to the same plane, and so on.

Now, let us assume any α elements of the principal diagonal plane of the given cubical determinant to be equal to zero and let us replace each of the corresponding variables by the negative of the element, placing the remaining $n^2-\alpha$ variables equal to zero; the general law may then be stated as follows:

A devertebrated cubical determinant of order n containing α zero elements in its principal diagonal plane may be developed into a series of terms of which the first is the given cubical determinant, $\triangle$; the next α being the products of each of the α elements into its co-planar cubical minor of order $n-1$; the next set consisting of the products of every possible pair of the α elements of the principal diagonal plane and its coplanar cubical minor of order $n-2$, no two of the elements being common to a same plane, and so on; the signs of the respective terms are positive or negative according as the combinations of the α elements are even or odd.

By a proper selection of the α elements this general law may be seen to contain as a particular case the law for the expansion of a devertebrated cubical determinant in which the α zero elements lie along the principal diagonal of the ordinary determinant of the principal diagonal plane.

For $\alpha=n^2$, the expansion will be seen to contain, also, the law for the expression of an invertebrate coplanar minor in terms of the vertebrate coplanar minors and the elements of the principal diagonal plane.

As illustrations, we may assume a cubical determinant, $\triangle'_\alpha$ of order 3 in which the elements in parentheses designate the variables which are added to the elements of the principal diagonal plane. We will write the sheets under one another, thus—

$$\triangle'_\alpha = \left\{ \begin{array}{lll} \left\{ \begin{array}{lll} 111+(111) & 112 & 113 \\ 121 & 122+(122) & 123 \\ 131 & 132 & 133+(133) \end{array} \right. & & \\ \left\{ \begin{array}{lll} 211+(211) & 212 & 213 \\ 221 & 222+(222) & 223 \\ 231 & 232 & 233+(233) \end{array} \right. & & \\ \left\{ \begin{array}{lll} 311+(311) & 312 & 313 \\ 321 & 322+(322) & 323 \\ 331 & 332 & 333+(333) \end{array} \right. & & \end{array} \right.$$

Expanding $\triangle'_\alpha$ as a cubical determinant with binomial elements, the expansion may be written in the following form—

$$\begin{aligned} \triangle'_\alpha = \triangle\ &+(111)[111]+(122)[122]+(133)[133]+(211)[211]+(222)[222] \\ &+(233)[233]+(311)[311]+(322)[322]+(333)[333] \\ &+(111)(222)[111,\ 222]+(111)(322)[111,\ 322]+(211)(122) \end{aligned}$$

$$
\begin{aligned}
&[211, 122]+(211)(322)[211, 322]+(311)(122)[311, 122]\\
&+(311)(222)[311, 222]+(122)(233)[122, 233]+(122)(333)\\
&[122, 333]+(222)(333)[222, 333]+(222)(133)[222, 133]\\
&+(322)(133)[322, 133]+(322)(233)[322, 233]+(133)(211)\\
&[133, 211]+(133)(311)[133, 311]+(233)(111)[233, 111]\\
&+(233)(311)[233, 311]+(333)(111)[333, 111]+(333)(211)\\
&[333, 211]+(111)(222)(333)+(111)(322)(233)\\
&+(122)(211)(333)+(122)(233)(311)+(133)(211)(322)\\
&+(133)(222)(311).
\end{aligned}
$$

For $a=1$, assume $(111)=-111$, and place the remaining variables equal to zero, then

$$\Delta'_1=\Delta-111[111].$$

For $a=2$, assume $(111)=-111$, $(333)=-333$ and place the remaining variables equal to zero, then

$$\Delta'_2=\Delta-111[111]-333[333]+111.222.333.$$

For $a=3$, assume $(111)=-111$, $(211)=-(222)=-222$, and place the remaining variables equal to zero, then

$$\Delta'_3=\Delta-111[111]-211[211]-222[222]+111.222.333.$$

We may now obtain very readily a formula for the number of terms in a cubical determinant which are independent of the elements of the principal diagonal plane.

In the preceding article it has been shown that the expansion of a devertebrated determinant may be put in the following form—

$$
\begin{aligned}
\Delta_a=\Delta\ &+\Sigma(111)[111]+\Sigma(111)(222)[111, 222]\\
&+\Sigma(111)(222)(333)[111, 222, 333]+\ldots\ldots\\
&+\Sigma(111)(222)(333)\ldots\ldots aaa[111, 222, \ldots\ldots, aaa]\\
&+(111)(222)\ldots\ldots(nnn),
\end{aligned}
$$

no two elements of a product belonging to a same plane. Placing $a=n$, thereby making Δ_n an invertebrate cubical determinant, we have

$$
\begin{aligned}
\Delta_n=\Delta\ &-\Sigma 111[111]+\Sigma 111.222[111, 222]-\Sigma 111.222.333[111, 222, 333]\\
&+\ldots\ldots+(-1)^r\Sigma 111.222\ldots\ldots rrr[111, 222\ldots\ldots rrr]\\
&+\ldots\ldots(-1)^n 111.222.333\ldots\ldots nnn.
\end{aligned}
$$

Since there are $(n!)^2$ terms in the expansion of a cubical determinant of order n, the number of terms in the preceding invertebrate cubical determinant will be

$$\phi(n)=(n!)^2-n^2[(n-1)!]^2+\frac{n^2(n-1)^2}{1.2}[(n-2)!]+\ldots\ldots$$

$$-\frac{n^2(n-1)^2(n-2)^2}{1.2.3}[(n-3)!]+\ldots\ldots$$

$$+(-1)^r\frac{n^2(n-1)^2\ldots\ldots(n-r+1)^2}{1.2.3\ldots\ldots r}[(n-r)!]+\ldots\ldots$$

$$+(-1)^n\frac{n^2(n-1)^2\ldots\ldots3.2.1}{1.2.3\ldots\ldots n}.$$

Removing from each of the terms the common factor $[n!]^2$ and noticing that the first two terms are equal but of opposite sign, we have

$$\phi(n)=(n!)^2\left\{\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots\ldots(-1)^n\frac{1}{1.2.3\ldots\ldots n}\right\}.$$

CAYLEY'S EXPANSION THEOREM.

Any cubical determinant may be developed into a series of terms of which the first is obtained by changing into zero all the elements of the principal diagonal plane, the next n^2 by multiplying each of the elements of the principal diagonal plane by its minor in the cubical determinant and altering the principal diagonal plane of the minor as the given cubical determinant was altered; the next $\frac{n^2(n-1)^2}{2!}$ by multiplying each pair of elements of the principal diagonal plane, no two coming from a same set of planes, by its minor in the cubical determinant and altering the principal diagonal plane of the minor as the original cubical determinant was altered, etc., the last $n!$ terms being every possible product, n at a time, of the elements of the principal diagonal plane, no two being taken from a same set.

By changing into zero all the elements of the principal diagonal plane we delete exactly those terms of the cubical determinant which contain any of these elements. The product of an element of the principal diagonal plane and its minor in the given cubical determinant gives exactly those terms of the given cubical determinant which contain that element.

Changing into zero all the elements of the principal diagonal plane of the minor we obtain the sum of all the terms of the cubical determinants which contain that element and *only* that element.

The expansion gives, therefore, first, all the terms of the determinant involving no element of the principal diagonal plane of the given cubical determinant; secondly, all those involving *only* one element; thirdly, all those involving *only* two elements, etc.

The number of terms in the expansion which contain only one element will be the same as the number of elements in the principal diagonal plane, or n^2.

Since in forming the pair of elements one and only one element of a pair can be taken from the same set of planes, it follows that $n(n-1)$ pairs can be taken from a same two sets of planes in which a corresponding pair of planes in the set are kept fixed and, therefore, in every possible way there can be formed

$$n(n-1)n_2 \text{ or } \frac{n^2\,(n-1)^2}{2!}$$

by choosing in every possible manner the pair of planes to be kept fixed.

It should be noticed that the expansion passes directly from those terms involving only $n-2$ elements of the principal diagonal plane to those involving them all. There would be no difficulty in stating at once the general law for the expansion of a cubical determinant in terms of the elements of the principal diagonal of the principal diagonal plane of the cubical determinant and their coaxial cubical minors.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

256. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Three men, A, B, and C, rented a pasture for a fixed amount, each to pay per month in proportion to the stock pastured. During the first month A put in 3 horses and B and C each some horses, and B paid for the month \$6, but A and C each defaulted payment. During the next month each put in one more horse, and C paid for the month \$7.20, but A and B each defaulted payment. During the next month each put in one more horse, and A paid his bill for the month, \$5, but B and C each defaulted.

Required: (1) the rent of the pasture per month; (2) the number of horses B and C each put in during the first month; and (3) the amount A, B, and C, each, owed for the unpaid service.

I. Solution by S. A. COREY, Hiteman, Iowa.

Let a, b, c=rate per horse per month for the first, second, and third month, respectively. Let x, y=number of horses put in the first month by B, and C, respectively. Let $n=3+x+y$=total number of horses in pasture first month. Let m=fixed monthly rental of pasture.

Then as A paid \$5 for the third month's rental when he had 5 horses in the pasture, c=\$1. As fewer horses were in the pasture the two preceding months the rate per horse per month was more than \$1 for the first and second months,

or both a and $b>\$1$. Then as x is an integer, as $a>\$1$, and as B paid but \$6 for the first month's service, $x \leqq 5$. Similarly, as y is an integer, $b>\$1$, and as C paid but \$7.20 for the second month's service $(y+1) \leqq 7$, or $y \leqq 6$. Hence as $n=3+x+y$, $n \leqq 14$. But as $c=m/(n+6)=1$, $m=(n+6)$, whence $m \leqq \$20$. But as not less than 5 horses were placed in the pasture the first month, $m=\geqq 11$.

B had x horses in the pasture the first month, the rate then being m/n, or $m/(m-6)$, and paid \$6 for the service. Hence, $xm/(m-6)=36$, or reducing and transposing, $(6-x)m=36$, but as $(6-x)=$integer, and $11 \leqq m \leqq 20$ (remembering that C had some horses in the pasture the first month, and that $n=m-6$). $6-x=2$, $m=\$18$, whence $x=4$ and $n=12$. But $y=n-3-x=5$. As $a=m/n=\$1.50$, $b=m/(n+3)=\$1.20$, and $c=\$1$, we readily find that

A owed \$4.50 for the first, and \$4.80 for the second month's service=\$9.30 total.
B owed \$6 for the second, and \$6 for the third month's service=\$12 total.
C owed \$7.50 for the first, and \$7 for the third month's service=\$14.50 total.

II. Solution by REV. J. H. MEYER, S. J., Professor of Mathematics, College of the Sacred Heart, Augusta, Ga.; by M. R. BECK, Cleveland, Ohio, and by J. EDWARD SANDERS, Reinersville, Ohio.

Using r as the number of dollars in one month's rent; x and y as the number of horses B and C put in at first, respectively, we obtain

$$\frac{rx}{3+x+y}=6, \text{ or } r=\frac{18+6x+6y}{x}. \tag{1}$$

$$\frac{r(y+1)}{6+x+y}=7.20, \text{ or } r=\frac{43.2+7.2x+7.2y}{y+1}. \tag{2}$$

$$\frac{5r}{9+x+y}=5, \text{ or } r=\frac{45+5x+5y}{5}. \tag{3}$$

Equating (1) and (3), $90-15x+30y=5x^2+5xy$ ..... (4). Equating (2) and (3), $171+31x-14y=5xy+5y^2$ ..... (5). Adding (4) and (5), $-5(x+y)^2+16(x+y)=-261$ ..... (6). Solving (6) for $x+y$, $x+y=9$ ..... (7). Substituting (7) in (3), (2), (1), $r=18$, $y=5$, $x=4$. For first month, \$1.50, for the second, \$1.20, and for the third, \$1.00 must be paid for each horse.

Hence, A owed \$9.30; B, \$12.00; C, \$14.50.

Also solved by O. L. Callecot, A. H. Holmes, L. E. Newcomb, J. Scheffer, G. B. M. Zerr, and the Proposer.

257. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x+y=10$, (2) $3x=\log_{10}y$.

I. Solution by HENRY HEATON, Belfield, N. D.

We have $3x=\log_{10}(10-x)$, or $10-x=10^{3x}$. If we suppose $x=\frac{1}{3}$, we obtain $9\frac{2}{3}=10$. An error of $\frac{1}{3}$. If we suppose $x=\frac{1}{4}$, we get $9.25=4.38$. An error of 4.87. By position we obtain $x=.328$. Substituting this we get $9.672=9.639$. An error of $-.043$. Substituting .329 for x we get $9.671=9.704$. An error of

.033. By position the second time we have $x=.3285$. Substituting this, we obtain $9.6715=9.6716$. An error of .0001.

Hence $x=.3285-$, and $y=10.-3285=9.6715+$.

II. Solution by A. H. HOLMES, Brunswick, Maine, and by J. SCHEFFER, Hagerstown, Md.

$x+y=10$ .....(1), $3x=\log_{10}y$ .....(2).

$\therefore \log y=\log(10-x)$ and $\log_{10}y=M\log(10-x)$. $\therefore \log(10-x)=3x/M$.

$$\log(10-x)=\frac{1}{M}+\log\left(1-\frac{x}{10}\right)=\frac{1}{M}-\frac{x}{10}-\frac{x^2}{200}-\frac{x^3}{3000}-\text{ etc.}=\frac{3x}{M}.$$

The series converges very rapidly and the first two terms are sufficient for a correct value of x to the fourth decimal figure.

Hence $\dfrac{1}{.434294}-\dfrac{x}{10}-\dfrac{x^2}{200}=\dfrac{3x}{.434294}$.

$\therefore x=.3285-$, and from (1), $y=9.6714+$.

III. Solution by S. A. COREY, Hiteman, Iowa.

From (1), $y=10-x$, and substituting in (2), $\log_{10}(10-x)=3x$.....(3), whence $\log_{10}[10(1-\frac{x}{10})]=3x$.....(4), and $\log_{10}10+\log_{10}(1-\frac{x}{10})=3x$.....(5). $\log_{10}(1-\frac{x}{10})=3x-1$.....(6). But as $\log_{10}(1-\frac{x}{10})<0$, we have $3x<1$. Let $3x=1-3v$. By substituting in (6), $\log_{10}(\frac{29}{30}+\frac{v}{10})=-3v$, or $\log_{10}[\frac{29}{30}(1+\frac{3v}{29})]=-3v$, $\log_{10}(1+\frac{3v}{29})=\log_{10}30-\log_{10}29-3v=a-3v$ ($a=.014,723,256,8$).....(7). But $\log_{10}(1+\frac{3v}{29})>0$, hence $3v<a$. Next let $3v=a-3w$. By substituting in (7), $\log_{10}\left(\frac{29+a}{29}-\frac{3w}{29}\right)=3w$, $\log_{10}\left[\left(\frac{29+a}{29}\right)\left(1-\frac{3w}{29+a}\right)\right]=3w$. Whence $\log_{10}\left(1-\frac{3w}{29+a}\right)=3w+\log_{10}29-\log_{10}(29+a)=3w-.000,220,434,6$.....(8). But $\log_{10}\left(1-\frac{3w}{29+a}\right)=\frac{-3mw}{29+a}$ nearly, where m=modulus of common system of logarithms, and therefore (8) becomes $\frac{3mw}{29+a}+3w=.000,220,434,6$, nearly.

Whence, $w=.000,072,394,6$, nearly; and $x=\frac{1}{3}-a/3+w=.328,497,975,7$, $y=10-x=9.671,502,024,3$.

Also solved by G. W. Greenwood, G. B. M. Zerr, and the Proposer.

258. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum the infinite series $\frac{n^2}{(4n^2-1)^2}$ beginning with $n=1$, n being always odd.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n^2}{(4n^2-1)^2}=\tfrac{1}{16}\left[\frac{1}{(2n-1)^2}+\frac{1}{(2n+1)^2}-\frac{1}{2n+1}+\frac{1}{2n-1}\right]$$

Let $u=1, 3, 5, 7, \ldots\ldots$ Then

$$\Sigma\frac{n^2}{(4n^2-1)^2}=\tfrac{1}{16}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\ldots\ldots\right]+\tfrac{1}{16}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\right.$$

$$\left.+\frac{1}{9}-\ldots\ldots\right].$$

$$\therefore\ \Sigma\frac{n^2}{(4n^2-1)^2}=\tfrac{1}{16}\cdot\frac{\pi^2}{8}+\tfrac{1}{16}\cdot\frac{\pi}{4}=\frac{\pi}{64}\left[\tfrac{1}{2}\pi+1\right]\quad\text{(See line 1, p. 41).}$$

Also solved by G. W. Greenwood, and Henry Heaton.

CALCULUS.

133. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Evaluate $\int\frac{\sqrt{(1+y)}}{1+y^2}dy$.

Solution by HENRY HEATON, Belfield, N. D.

Let $1+y=z^2$; then $\int\frac{\sqrt{(1+y)}}{1+y^2}dy=\int\frac{2z^2dz}{z^4-2z^2+2}=\frac{1}{2a}\int\left(\frac{z}{z^2-2az+\sqrt{2}}\right.$

$$\left.-\frac{z}{z^2+2az+\sqrt{2}}\right)dz,\ \text{where}\ a=\sqrt{\frac{\sqrt{2}+1}{2}}.$$

$$\frac{1}{2a}\int\left(\frac{z}{z^2-2az+\sqrt{2}}-\frac{z}{z^2+2az+\sqrt{2}}\right)dz=\frac{1}{2a}\int\left(\frac{(z-a)+a}{z^2-2az+\sqrt{2}}\right.$$

$$\left.\frac{(z+a)-a}{z^2-2az+\sqrt{2}}\right)dz=\frac{1}{2a}\log\sqrt{\frac{z^2-2az+\sqrt{2}}{z^2+2az+\sqrt{2}}}+\frac{1}{2\sqrt{[\sqrt{2}-a^2]}}\left(\tan^{-1}\frac{z-a}{\sqrt{[\sqrt{2}-a^2]}}\right.$$

$$\left.+\tan^{-1}\frac{z+a}{\sqrt{[\sqrt{2}-a^2]}}\right)=\sqrt{\frac{\sqrt{2}-1}{2}}\log\sqrt{\frac{1+y-\sqrt{[2\sqrt{2}+2]}\sqrt{[1+y]}+\sqrt{2}}{1+y+\sqrt{[2\sqrt{2}+2]}\sqrt{[1+y]}+\sqrt{2}}}$$

$$+\sqrt{\frac{\sqrt{2}+1}{2}}\tan^{-1}\left(\frac{\sqrt{[2\sqrt{2}-2]}\sqrt{[1+y]}}{\sqrt{2}-1-y}\right).$$

The above solution is in the real factors of z^4-2z^2+2.

On page 140, No. 5, Vol. IX, Dr. Zerr gives a solution, using the imaginary factors of z^4-2z^2+2. His final seemingly imaginary result reduces to a real value by substituting $\sqrt{\frac{1+\sqrt{2}}{2}}+\sqrt{\frac{1-\sqrt{2}}{2}}$ for $\sqrt{[1+\sqrt{(-1)}]}$ and $\sqrt{\frac{1+\sqrt{2}}{2}}-\sqrt{\frac{1-\sqrt{2}}{2}}$ for $\sqrt{[1-\sqrt{(-1)}]}$, reducing and remembering that

$$\log[a+b\sqrt{(-1)}]=\sqrt{(-1)}\tan^{-1}\frac{b}{a}+\log\sqrt{(a^2+b^2)}.$$

216. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Find the limit of the sum of the series

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{n}{n^2+m^2},$$

when n and m are indefinitely increased. (Distinguish the several cases arising from the different *relative* values of m and n.)

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{n}{n^2+m^2}$$

$$=h\left[1+\frac{1}{1+h^2}+\frac{1}{1+(2h)^2}+\frac{1}{1+(3h)^2}+\ldots+\frac{1}{1+(mh)^2}\right]$$

where $h=\frac{1}{n}$. Hence the required limit is $\int_0^{mh}\frac{dx}{1+x^2}=\tan^{-1}(mh)$.

If m is small compared with n so that $mh=0$, the limit is $\tan^{-1}0=0$. If m is equal to n so that $mh=1$, the limit is $\tan^{-1}1=\frac{1}{4}\pi$. If m is large compared with n so that $mh=\infty$, the limit is $\tan^{-1}\infty=\frac{1}{2}\pi$.

Also solved by S. A. Corey, and Henry Heaton.

217. Proposed by S. A. COREY, Hiteman, Iowa.

In *The Analyst*, Vol. II, p. 120, 1875, Dr. G. W. Hill finds by the method of mechanical quadrature the value of $\int_0^{\frac{1}{2}\pi}\frac{xdx}{\sin x[1+.16\cos^2 x]^{\frac{1}{2}}}$ to be 1.6576363.

Evaluate the definite integral by some other method and verify above result.

Solution by the PROPOSER.

Using the formula* $f(x)-f(0)=\frac{x}{m.2}\left\{[f'(x)+f'(0)]+2\left[f'\left(\frac{x}{m}\right)\right.\right.$

MONTHLY, Vol. 13, p. 12.

$$+f'\left(\frac{2x}{m}\right)+f'\left(\frac{3x}{m}\right)+\ldots\ldots+f'\left(\frac{m-1}{m}x\right)\Big]\Big\}-\frac{B_1x^2}{m^2.2!}[f'(x)-f'(0)]$$

$$+\frac{B_2x^4}{m^4.4!}[f^{iv}(x)-f^{iv}(0)]-\ldots\ldots+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}[f^{(2n)}(x)-f^{(2n)}(0)]+\ldots\ldots$$

(B_1, B_2, ..., B_n being Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc.) and taking $m=6$, we get

$\frac{1}{24}\pi f'(\frac{1}{2}\pi)=\pi^2/48=$ 205,616,758

$\frac{1}{24}\pi f'(0)=\dfrac{\pi}{24(1.16)^{\frac{1}{2}}}=$ 104,773,547

$\frac{1}{12}\pi f'(\frac{1}{12}\pi)=\dfrac{\pi^2}{144\sin 15^\circ(1+.16\cos^2 15^\circ)^{\frac{1}{2}}}=$ 214,932,009

$\frac{1}{12}\pi f'(\frac{1}{6}\pi)=$ 231,297,112

$\frac{1}{12}\pi f'(\frac{1}{4}\pi)=$ 259,082,377

$\frac{1}{12}\pi f'(\frac{1}{3}\pi)=$ 298,480,940

$\frac{1}{12}\pi f'(\frac{5}{12}\pi)=$ 349,155,189

Sum $=1.663,337,932$

$\dfrac{B_1(\frac{1}{2}\pi)^2}{6^2.2!}[f'(\frac{1}{2}\pi)-f'(0)]=\dfrac{\pi^2}{1728}=$ 005,711,577

$1.657,626,355$

$\dfrac{B_2(\frac{1}{2}\pi)^4}{6^4.4!}[f^{iv}(\frac{1}{2}\pi)-f^{iv}(0)]=\dfrac{\pi^4}{30.12^4.4!}[3-9(.16)]=\dfrac{156\pi^4}{3000.12^4.4!}=$ .000,010,178

$1.657,636,533$

$\dfrac{B_3(\frac{1}{2}\pi)^6}{6^6.6!}[f^{vi}(\frac{1}{2}\pi)-f^{vi}(0)]=\dfrac{\pi^6}{42.12^6.6!}[225(.16)^2-30(.16)+25]$

$=\dfrac{2596\pi^6}{4200.12^6.6!}=$ 000,000,009

Value of integral correct to eight or 9 decimal places . $1.657,636,524$

Value of integral obtained by Hill . . . . . $1.657,636,3$

As Hill used but seven decimal places in his computations we would not expect his result to be correct to the last decimal place. In the foregoing computations the term involving B_3 might have been omitted and the result would still have been more accurate than Hill's result, although his method involves somewhat more labor than does the foregoing.

219. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

In the article "Infinitesimal Calculus" in the *Encyclopaedia Britannica* Vol. XIII, page 24, I notice the following: "Of all triangular pyramids standing on a given triangular base, and of given altitude, find that whose surface is the least." A solution is required.

I. Solution by the PROPOSER.

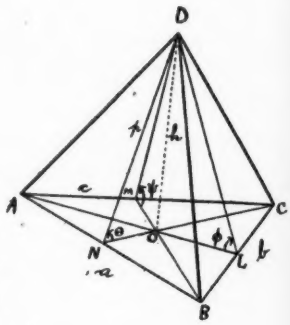
Let ABC be the given triangular base, whose sides are a, b, c, and let h be the given altitude; also let θ, ϕ, ψ, be the angles of inclination of the faces of the pyramid to the base. Now, in the face DAB, if p is the perpendicular from the vertex on the side AB, then $\sin\theta = h/p$, hence $p = h \operatorname{cosec}\theta$.

But area of $\triangle DAB = \frac{1}{2}ap = \frac{1}{2}ah \operatorname{cosec}\theta$. Hence, the area of the *three* faces of the pyramid is

$$u = \tfrac{1}{2}h(a \operatorname{cosec}\theta + b \operatorname{cosec}\phi + c \operatorname{cosec}\psi) \ldots\ldots (1).$$

Also, the base of the pyramid may be divided into three triangles, whose altitudes are easily found. For

since $\dfrac{h}{\text{altitude of } \triangle AOB} = \tan\theta$.



$\therefore$ Altitude of $\triangle AOB = h \cot\theta$.

$\therefore$ Area of $\triangle AOB = \frac{1}{2}ah \cot\theta$.

Hence the area of the *whole* base ABC is

$$v = \tfrac{1}{2}h(a \cot\theta + b \cot\phi + c \cot\psi) \ldots\ldots (2).$$

Now, equations (1) and (2) are perfectly general expressions for the surface and base of a pyramid with the given dimensions, and apply to any figure besides the one I have given. Now, from (1),

$$\frac{\partial u}{\partial\theta} = \frac{h}{2}\left[-a \operatorname{cosec}\theta \cot\theta - c \operatorname{cosec}\psi \cot\psi \frac{\partial\psi}{\partial\theta}\right] = 0.$$

$$\frac{\partial u}{\partial\phi} = \frac{h}{2}\left[-b \operatorname{cosec}\phi \cot\phi - c \operatorname{cosec}\psi \cot\psi \frac{\partial\psi}{\partial\phi}\right] = 0.$$

Whence, $a \operatorname{cosec}\theta \cot\theta \dfrac{\partial\psi}{\partial\phi} = -c \operatorname{cosec}\psi \cot\psi \dfrac{\partial\psi}{\partial\theta} \cdot \dfrac{\partial\psi}{\partial\phi}$,

and $b \operatorname{cosec}\phi \cot\phi \dfrac{\partial\psi}{\partial\theta} = -c \operatorname{cosec}\psi \cot\psi \dfrac{\partial\psi}{\partial\phi} \cdot \dfrac{\partial\psi}{\partial\theta}$.

$$\therefore a \operatorname{cosec}\theta \cot\theta \frac{\partial\psi}{\partial\phi} = b \operatorname{cosec}\phi \cot\phi \frac{\partial\psi}{\partial\theta} \ldots\ldots (3).$$

From (2), $\dfrac{2v}{h} = a \cot\theta + b \cot\phi + c \cot\psi$. $\therefore c \cot\psi = \dfrac{2v}{h} - a \cot\theta - b \cot\phi$.

Differentiating successively as to θ and ϕ, we have

$$-c\ \text{cosec}^2\ \psi\ \frac{\partial\psi}{\partial\theta}=a\ \text{cosec}^2\ \theta,\ \text{and}\ -c\ \text{cosec}^2\ \psi\frac{\partial\psi}{\partial\phi}=b\ \text{cosec}^2\ \phi.$$

Whence, $\frac{\partial\psi}{\partial\theta}=-\frac{a\ \text{cosec}^2\ \theta}{c\ \text{cosec}^2\ \psi}$, $\frac{\partial\psi}{\partial\phi}=-\frac{b\ \text{cosec}^2\ \phi}{c\ \text{cosec}^2\ \psi}$.

Substituting these values in (3), we get

$$a\ \text{cosec}\ \theta\ \cot\ \theta\ \frac{b\ \text{cosec}^2\ \phi}{c\ \text{cosec}^2\ \psi}=b\ \text{cosec}\ \phi\ \cot\ \phi\ \frac{a\ \text{cosec}^2\ \theta}{a\ \text{cosec}^2\ \psi}.$$

$\therefore \cot\ \theta\ \text{cosec}\ \phi=\cot\ \phi\ \text{cosec}\ \theta.$ $\cos\ \theta=\cos\ \phi$; $\therefore \theta=\phi$.

Similarly, by finding $\frac{\partial u}{\partial\theta}$, $\frac{\partial u}{\partial\psi}$, it may be shown that $\theta=\psi$. Hence, $\theta=\phi=\psi$, and the faces of the pyramid must be equally inclined to the base.

As it is evident from the nature of the problem that there is a minimum, it is unnecessary to proceed to the higher derivatives.

The results can be generalized to apply to a pyramid of any number of faces, but the work is a little more complicated.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC represent the given base, and $DO=h$ the given altitude. In the base ABC, from O let fall the three perpendiculars x, y, z upon the sides BC, AC, and AB, respectively. Then the lateral surface will be $=\frac{1}{2}a\sqrt{(h^2+x^2)}+\frac{1}{2}b\sqrt{(h^2+y^2)}+\frac{1}{2}c\sqrt{(h^2+z^2)}$.

Consequently, $M=a\sqrt{(h^2+x^2)}+b\sqrt{(h^2+y^2)}+c\sqrt{(h^2+z^2)}$ is to be a minimum, subject to the relation $ax+by+cz=2\triangle$, $\triangle$ denoting the area of $\triangle ABC$. From the first of these equations we get, by differentiation

$$\frac{\partial M}{\partial x}=\frac{ax}{\sqrt{(h^2+x^2)}}+\frac{cz}{\sqrt{(h^2+y^2)}}\frac{\partial z}{\partial x},\quad \frac{\partial M}{\partial y}=\frac{by}{\sqrt{(h^2+y^2)}}+\frac{cz}{\sqrt{(h^2+z^2)}}\frac{\partial z}{\partial y},$$

and from the second we get

$$\frac{\partial z}{\partial x}=-\frac{a}{c},\quad \frac{\partial z}{\partial y}=-\frac{b}{c}.$$

Substituting, and putting $\frac{\partial M}{\partial x}$, $\frac{\partial M}{\partial y}$ each $=0$, we obtain

$$\frac{x}{\sqrt{(h^2+x^2)}}=\frac{z}{\sqrt{(h^2+z^2)}}=\frac{y}{\sqrt{(h^2+y^2)}}.$$

Hence $x=y=z$. The pyramid of a minimum surface is therefore the one whose faces are equally inclined to the base.

DIOPHANTINE ANALYSIS.

133. Proposed by REV. R. D. CARMICHAEL, Hartselle, Alabama.

Find all perfect numbers of four primes and of multiplicity 4.

Solution by the PROPOSER.

The object of this note is to show that $2^5.3^3.5.7=30240$ is the only multiply perfect number* of multiplicity 4 and having only 4 distinct primes.

Let $m=p_1^{a_1}p_2^{a_2}p_3^{a_3}p_4^{a_4}$ be the number where the primes p_1, p_2, p_3, p_4, are in order of magnitude, beginning with the smallest. The sum of the divisors is equal to four times the number. Dividing this equation by the number, have

$$4=\frac{p_1^{a_1+1}-1}{p_1^{a_1}(p_1-1)}\cdot\frac{p_2^{a_2+1}-1}{p_2^{a_2}(p_2-1)}\cdot\frac{p_3^{a_3+1}-1}{p_3^{a_3}(p_3-1)}\cdot\frac{p_4^{a_4+1}-1}{p_4^{a_4}(p_4-1)}\quad\ldots\ldots(1).$$

$$\therefore 4<\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\cdot\frac{p_3}{p_3-1}\cdot\frac{p_4}{p_4-1}\quad\ldots\ldots(2),$$ from which it may easily be shown that $p_1=2$, $p_2=3$, $p_3=5$, and $p_4=7$, 11, or 13. From (1) we may write (for use later)

$$4<\frac{2^{a_1+1}-1}{2^{a_1}}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{p_4}{p_4-1}\quad\ldots\ldots(3).$$

Again from (1),

$$4=\frac{2^{a_1+1}-1}{2^{a_1}}\cdot\frac{3^{a_2+1}-1}{3^{a_2}.2}\cdot\frac{5^{a_3+1}-1}{5^{a_3}.4}\cdot\frac{p_4^{a_4+1}-1}{p_4^{a_4}(p_4-1)}\quad\ldots\ldots(4).$$

$$\therefore 2^{a_1+5}.3^{a_2}.5^{a_3}.p_4^{a_4}(p_4-1)=(2^{a_1+1}-1)(3^{a_2+1}-1)(5^{a_3+1}-1)(p_4^{a_4+1}-1)\quad\ldots..(5).$$

Obviously n may be so taken that

$$2^{a_1+1}-1=(2^{2n+1}-1)(2^{2n+1}+1)(2^{2(2n+1)}+1)(2^{4(2n+1)}+1)\quad\ldots\ldots,\quad\ldots\ldots(6)$$

where limitations are to be found for the series of factors. The fourth factor introduces the prime 17, and hence not more than 3 factors may be used. We may easily show that $2^{2n+1}-1$ is not divisible by 3 or 5; and also that it is not a power of $p_4=11$ or 13; and that if it is a power of $p_4=7$, it is the first power. Hence, if $p_4=11$ or 13, $n=0$, and we have from equation (6), $a_1=1$ or 3. These values of p_4 and a_1 will not satisfy equation (3), however they are combined. Now, if $p_4=7$, $n=0$ or 1 (by the preceding). The possible values of a_1 are found from (6) to be $a_1=1, 2, 3, 5, 7$. When $a_1=7$ the prime 127 is introduced, and this value must therefore be discarded. We therefore have left to consider the cases of $p_4=7$ and $a_1=1, 2, 3$, or 5. Substituting $p_4=7$ in equation (5) we have

*This term was introduced by D. N. Lehmer in 1901; see *Annals of Mathematics*, Ser. 2, Vol. 2, p. 103. "A multiply perfect number is one which is an exact divisor of the sum of all the divisors, the quotient being the multiplicity."

$$2^{a_1+6}.3^{a_2+1}.5^{a_3}.7^{a_4}=(2^{a_1+1}-1)(3^{a_2+1}-1)(5^{a_3+1}-1)(7^{a_4+1}-1) \quad \ldots\ldots\ldots(7).$$

Now n may be so taken that

$$3^{a_2+1}-1=(3^{2n+1}-1)(3^{2n+1}+1)(3^{2(2n+1)}+1)(3^{4(2n+1)}+1) \quad \ldots\ldots\ldots, \quad \ldots\ldots\ldots(8).$$

The last factor written introduces the prime 41, and therefore not more than three factors can be considered. It may easily be shown that $3^{2n+1}-1$ is not divisible by 5 or 7, and that it contains the factor 2 but once. Therefore, $3^{2n+1}-1=2$. Hence, $n=0$. Equation (8) now yields $a_2=1$ or 3.

A trial of each of the eight possible cases produced by every possible combination of the values $a_2=1$ or 3, and $a_1=1$, 2, 3, or 5, will result in finding but one number of the type here considered, namely, $2^5.3^3.5.7$.

MECHANICS.

187. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Find the path described by a particle acted upon by a central force, the force being directly proportional to the distance of the particle.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Take coördinate axes through the center of force, and let μ^2 be the force on a particle of unit mass at the unit distance. Then the equations of motion are

$$\frac{d^2x}{dt^2}=-\mu^2 r\cos\theta=-\mu^2 x, \quad \frac{d^2y}{dt^2}=-\mu^2 r\sin\theta=-\mu^2 y.$$

Integrating the equations of motion we have

$$x=a\cos\mu t+b\sin\mu t, \quad y=c\cos\mu t+d\sin\mu t.$$

$\therefore (cx-ay)^2+(dx-by)^2=(bc-ad)^2$, which is an ellipse.

Also solved by S. A. Corey, and Henry Heaton.

PROBLEMS FOR SOLUTION.

ALGEBRA.

262. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\frac{n}{(4n^2-1)^2}$, beginning with $n=1$, n being always odd.

263. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the transcendentals e and π in the form of infinite continued fractions.

264. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the invariant $2(a_0a_4-4a_1a_3+3a_2^2)$ of the binary quartic $a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2+4a_3x_1x_2^3+a_4x_2^4$ in terms of roots of the latter.

AVERAGE AND PROBABILITY.

177. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random planes cut a given sphere. What is the chance that they intersect within the sphere?

CALCULUS.

220. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

To determine the least polygon of n sides that can be described about a given circle.

DIOPHANTINE ANALYSIS.

135. Proposed by A. H. HOLMES, Brunswick, Maine.

In the equation in Diophantine Analysis: $2x^2+2x+1=\square=u^2$, show that u is always the sum of two squares.

136. Proposed by A. H. HOLMES, Brunswick, Maine.

Given $7x^2-111=y^2$. Required a value for y greater than unity which shall be a prime integer.

GEOMETRY.

288. Proposed by C. N. SCHMALL, College of the City of New York, New York City.

From a point P on a given circle to draw two chords such that, (α) chord PA : chord $PB=m:n$ (a given ratio), and, (β) arc PA : arc $PB=1:3$.

289. Proposed by J. J. QUINN, Ph. D., Warren, Pa.

(a) Suppose a circle described around the origin. Then at the end of a uniformly revolving radius r, a line equal to the diameter is pivoted. Find the equation of the locus of its extremity, if for every unit of angle its projection on the X axis is a constant linear unit, being the same part of the diameter as the angle is of π radians.

(b) Show how it can be applied to the trisection or multisection of an angle.

290. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Show that the point (1, 1) is a conjugate point on the locus $x^3+y^3-3xy+1=0$.

MISCELLANEOUS.

158. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

In ingot of pure gold was melted at the Mint and then 10 ounces were taken out and 10 ounces of pure silver added and the contents of the melting pot mixed thoroughly. This was repeated until there were 10 such operations in all. The contents of the pot being then assayed was found to be nine-tenths fine, or standard gold. What was the weight of the original ingot? There was no loss in the precious metals by the melting.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Diophantine Analysis, No. 132. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

From the numbers, 0, 1, 2, ..., 42, select seven, such that the 42 differences of these seven numbers shall be congruent (mod. 43) to the numbers 0, 1, 2, ..., 42. The differences may be both positive and negative.*

Mechanics, No. 188. Proposed by H. L. ORCHARD, M. A., B. Sc. (Unsolved problem in Educational Times, London.)

Spherical bubbles are rising in water. Find the relation between radius and velocity.

NOTES AND NEWS.

Professor J. H. Jeans of Princeton University, has been elected fellow of the Royal Society of London.

Dr. E. B. Wilson has been promoted to an assistant professorship of mathematics at Yale University.

Dr. Oliver E. Glenn has been appointed instructor in mathematics in the University of Pennsylvania.

Professor W. J. Hussey, of Lick Observatory, has been appointed professor of astronomy at the University of Michigan.

*These problems involve important principles. Solutions have been contributed, but all incorrect. Will some reader make a study of them? ED. G.

The death is announced of Professor James Mills Peirce, Perkins Professor of Mathematics and Astronomy at Harvard University.

Mr. Louis A. Martin, Jr., has been promoted from instructor to assistant professor of mathematics and mechanics in Stevens Institute of Technology.

The April number of *Annals of Mathematics* contains "Theory of Fourier's Series" (concluded), by M. Bôcher; "Note on Multiply Perfect Numbers," by R. D. Carmichael; "Note on Integrating Factors," by Edwin Bidwell Wilson.

The latter part of Dr. Halsted's essay "The Value of Non-Euclidean Geometry" (*Popular Science Monthly*, November, 1905) has been reprinted in New Zealand with added notes by F. W. Frankland [Okataina, Foxton, Manawatu, New Zealand].

The American Association for the Advancement of Science will hold a summer session at Ithaca, New York, during the week June 26 to July 2. Dr. Edward Kasner is vice president, and Professor L. G. Weld secretary of Section A, Mathematics and Astronomy.

Dr. S. T. Tamura, mathematician in the department of terrestrial magnetism of the Carnegie Institution, has been offered a professorship of dynamics and ship's magnetism in the Naval Staff College, Tokyo, which is the graduate school for Japanese naval officers.

The one hundred and twenty-eighth regular meeting of the American Mathematical Society was held at Columbia University, New York City, on April 28th. The nineteenth meeting of the Chicago Section of the Society convened at Northwestern University, on April 14th. A total of thirty-nine papers were presented at these meetings.

The Thirteenth Summer Meeting and Fifth Colloquium of the American Mathematical Society will be held at Yale University during the entire week of September 3—8, 1906. The Colloquium which will open on Wednesday morning, will include the following courses of lectures: Professor E. H. Moore, "On the theory of bilinear functional operators;" Professor Max Mason, "Selected topics in the theory of boundary value problems of differential equations;" Professor E. J. Wilczynski, "Projective differential geometry."

JAMES MILLS PEIRCE. The Faculty of Arts and Sciences of Harvard University have adopted the following minutes on the life and services of the late Professor Peirce:

"The Faculty of Arts and Sciences desire to put on record their sense of great loss which they have sustained in the death of Professor James Mills Peirce.

"Born in Cambridge within sound of the College bell, a member of the faculty of Harvard College at twenty, serving for nearly fifty years, not as teacher merely, but successively as secretary of the academic council, as dean—

and almost as father—of the graduate school, and as dean of the faculty of arts and sciences, he spent the whole of a long life in and for the University.

"He was an admirable teacher, steeped in his subject, not buried in it, and always in close sympathy with his students, to whom he was ever a generous and inspiring friend. Broad minded and many sided, his scholarship was of that wide, human kind which unites learning with recognition of every accomplishment of grace of life, with interest in every intellectual problem, and with good will to every earnest man. All his work was characterized by thoroughness and finish, and by a kind of fervid loyalty. He had a high and large conception of academic freedom, and, in age as in youth, he looked forward and not back. Of a peculiarly lovable nature, courteous and kindly, he was known to all who met him for his friendly greeting, his earnest speech, at once measured and impetuous, and his scorn of anything narrow, or crooked, or mean."

At the University of Wisconsin, Dr. Edward B. VanVleck, now professor of mathematics at Wesleyan University, has been appointed to the professorship of mathematics left vacant by the resignation of Professor E. A. VanVelzer.

BOOK NOTICES.

Elementary Algebra. By G. A. Wentworth. Half Leather, vi+421 pages. Boston, New York, Chicago: Ginn & Co.

One of the best features of this book is a new set of exercises, some four thousand in number, with which the author has supplied the text. The appearance of the volume is attractive; the use of colors in plotting equations may be commended. The author shows a predilection for using more or less useless technical terms, "compound expressions" for polynomials, "scalar numbers" for real numbers, "orthotomics" for imaginaries. The real and imaginary axes are said to intersect "orthotomically." This thing should be avoided in an elementary algebra. The book would be more readable by student and teacher if divested of some inaccuracies. We read on page 1, "Whatever admits of increase or decrease is called *a magnitude*." The length of a meter stick for example! On page 289 the student learns that "Gravity always impresses upon a body, free to move, a downward velocity of g units each second, whether the body starts from a state of rest or is moving already with any velocity in any direction."

The treatment of Factoring is adequate, as is the graphical work in connection with simultaneous equations and imaginaries. But eleven pages are devoted to solving quadratics by completing the square, while the quadratic formula gets barely three pages, part fine print. The idea of a function is rather difficult for secondary students, but its introduction in the author's excellent section on variation may find justification. The section on "Laws of Physics" should not precede the treatment of variation. On the whole *Elementary Algebra* is not a contribution to mathematical pedagogy, but it is a teachable book and no doubt will meet with considerable success.

An Elementary Text-Book of Theoretical Mechanics. By George A. Merrill, B. S., Principal of the California School of Mechanical Arts, and Director of the Wilmerding School of Industrial Arts. Half Leather, 268 pages, 168 diagrams. San Francisco, New York, Cincinnati, Chicago: American Book Co.

This book is intended for upper classes in secondary schools and lower classes in colleges. It is almost alone in its field, so far as American publications are concerned. The teaching of Mechanics as a subject *per se* has been confined in the main to collegiate courses, and the few American text-books on the subject have been written for students familiar with the calculus. There is need of giving greater prominence to this subject in secondary schools, especially in institutions whose graduates look forward to industrial careers. Professor Merrill's book is most admirably suited to supply this need. The text is divided into three principal sections, Kinematics, Statics, and Kinetics. It is written "from the standpoint of the student," is clear, forceful, and pedagogical. Vectors are introduced at the start, and graphical methods and illustrations are used profusely and effectively, throughout the book. The attractiveness of the work is much enhanced by the use of several kinds of type, and by the large number of excellent illustrations. Good full page portraits of Galileo, Hugenius, and Newton adorn the work.

A Brief Introduction to the Infinitesimal Calculus. By Irving Fisher, Ph. D., Professor of Political Economy in Yale University. Second Edition. Cloth, 84 pages. London: Macmillan and Co.

The author states in his preface that this little book contains the substance of lectures by which he has been accustomed to introduce advanced students to a course in modern economic theory. Theprocesses of differentiation and integration are introduced and illustrated by elemeptary applications.

A First Course in Physics. By Robert Andrews Millikan, Ph. D., Assistant Professor of Physics in the University of Chicago, and Henry Gordon Gale, Ph. D., Instructor of Physics in the University of Chicago. Boston, New York, Chicago, London: Ginn & Co.

The advance pages of this new secondary school physics indicate that the authors are using the best efforts of scholarship to produce a text that will answer the demand of the day that the first course in physics be not only completely scientific, but be full of life and interest for the student as well. Judging from the first hundred pages of the book, clear and simple English, profuseness of illustration, and constant reference and application to actual natural phenomena are some of the strong features which are to characterize *A First Course in Physics.*

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ON THE n-SECTION OF AN ANGLE.

By REV. R. D. CARMICHAEL, Hartselle, Alabama.

The locus of the polar equation $r=a\cos\frac{n-1}{n}\theta$ is readily applicable to the problem of the n-section of angle. We plot a segment of the curve for $n=3$.

Let $OA=a$, and let O be the origin. Construct the circular segment AQM with O as center. The given locus and the circle coincide at A. Place the angle whose nth part is required in the position OAP, P a point on the given locus. Draw PQ perpendicular to OP. The angle AOQ is the required nth part of AOP. For, $OP=a\cos\frac{n-1}{n}\theta$, θ the angle AOP; hence $PQ=a\sin\frac{n-1}{n}\theta$; and therefore $\angle POQ=\frac{n-1}{n}\theta$; then, $\angle AOQ=\frac{\theta}{n}$. For the case constructed, $n=3$, this gives the trisection of the angle.

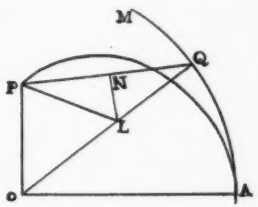

Fig. 1.

It is desirable to have some method of describing this curve by continuous motion. Let L in the above figure be the middle point of OQ. Then, since OPQ is a right angle, $PL=OL=LQ$. Draw LN perpendicular to PQ. It is parallel to OP. Therefore $\angle NLQ=\angle POQ=\frac{1}{2}\angle PLQ$. Hence, $\angle PLQ=\frac{2(n-1)}{n}\theta$. Now, let a material circle be fixed to PL with center at L. Also, let another circle